International Journal of Theoretical Physics, Vol. 35, No. 2, 1996

Three-Dimensional Representation of the Double Quantum Algebra $su_q(\eta(J))$ and the q-Deformation of the Double Complex Ernst Equation

Feng Lijun[1] **and Zhong Zaizhe**[1]

Received June 19, 1995

A three-dimensional representation of the double quantum algebra $su_q(\eta(J))$ is given. By the use of this representation and a Lax pair, we obtain a nonlinear Ernst equation system. By the harmonic function method, a solution of the q-deformed double complex Ernst equation is given.

1. INTRODUCTION

In recent years, the study of quantum groups has developed rapidly and many interesting results have been obtained, but the works which combines quantum groups with gravitational fields are few. Zhong (1992a) has linked the three-dimensional representation of the quantum algebra $su_q(2)$ with cylindrical gravitational waves. However, no solution was given. Zhong (1992b) also discussed the double quantum algebra $su_q(\eta(J))$, which we think must be linked with the gravitational field equations. In this paper, we obtain a nonlinear system which is in fact a q-deformation of the double complex Ernst equation. Furthermore, by the harmonic function method, we obtain a solution of this equation. This solution perhaps describes some aspects related to the q-deformation of gravitational fields.

The organization of this paper is as follows: In Section 2 we discuss the three-dimensional representation of the double quantum algebra $su_q(\eta(J))$. Section 3 gives the q-deformation of the double complex Ernst equation. In Section 4, by the harmonic function method, we give the solution of the q-deformed double complex Ernst equation.

[1] Department of Physics, Liaoning Normal University, Liaoning, Dalian 116029, China.

0020-7748/96/0200-0245$09.50/0

2. DOUBLE QUANTUM ALGEBRA $su_q(\eta(J))$

The general double complex function method has been discussed in several papers (Zhong, 1985, 1988, 1989). We use the relevant results. Let J denote the double-imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \epsilon$ ($\epsilon^2 = +1$, $\epsilon \neq \pm 1$). We take the generators as

$$K_+ = \begin{bmatrix} 0 & J^3 \\ 0 & \end{bmatrix}, \qquad K_- = \begin{bmatrix} & 0 \\ J & \end{bmatrix}, \qquad K_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \tag{1}$$

which obey the commutation relations

$$[K_3, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = 2K_3 \tag{2}$$

This means that K_+, K_- and K_3 generate the double Lie algebra $su(\eta(J))$, where the signature is

$$(\eta(J)) = \begin{bmatrix} 1 & \\ & -J^2 \end{bmatrix} \tag{3}$$

i.e., for $J = i$ it is $su(2)$, and for $J = \epsilon$ it is $su(1, 1)$.

The double quantum algebra $su_q(\eta(J))$ (Zhong, 1992), which is a q-deformation of $su_q(\eta(J))$, is generated by K_3^q and $K_\pm^q$ obeying the relations

$$[K_3^q, K_\pm^q] = \pm K_\pm^q, \qquad [K_+^q, K_-^q] = [2K_3^q]_q \tag{4}$$

where the quantum number is

$$[\chi]_q \equiv \frac{q^\chi - q^{-\chi}}{q - q^{-1}} = \frac{\sinh(\gamma\chi)}{\sinh \gamma}, \qquad \gamma = \ln q \tag{5}$$

It must be stressed that in this paper we only consider the case of a real parameter q, $q \in (0, 1]$; therefore $q + q^{-1} \geq 2$. On the real axis when $q \to 1$, then $K_3^q \to K_3$, $K_\pm^q \to K_\pm$. We are interested in the case of the three-dimensional representation (Biedenharn, 1989)

$$K_+^q = \begin{bmatrix} 0 & \sqrt{\Delta} & 0 \\ 0 & 0 & \sqrt{\Delta} \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{\Delta}{2}\right)^{1/2} K_+,$$

$$K_-^q = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{\Delta} & 0 & 0 \\ 0 & \sqrt{\Delta} & 0 \end{bmatrix} = \left(\frac{\Delta}{2}\right)^{1/2} K_-$$

$$K_3^q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = K_3 \tag{6}$$

where we write $[2]_q$ simply as Δ, $\Delta = q + q^{-1}$.

Generally, $su_q(\eta(J))$ does not correspond to a proper Lie algebra. However, the above representation is a special case, since it is easily seen that the linear mapping

$$\mathcal{A}: \quad su_q(\eta(J)) \to su(\eta(J))$$
$$\mathcal{A}(K^q_\pm) = \left(\frac{\Delta}{2}\right)^{1/2} K_\pm, \qquad \mathcal{A}(K^q_3) = K_3 \tag{7}$$

is an isomorphic mapping. This means that in the three-dimensional case with K^q_3, $K^q_\pm$ taken as in equation (6), $su_q(\eta(J))$ is isomorphic to the Lie algebra $su(\eta(J))$. In the following, we only discuss this case.

Now, we take the generating elements of $su_q(\eta(J))$ as

$$L^q_1 = \frac{J^3}{2} K^q_3, \qquad L^q_2 = \frac{1}{2J}(K^q_+ + K^q_-), \qquad L^q_3 = \frac{1}{2}(K^q_+ - K^q_-) \tag{8}$$

Then the commutation relations are

$$[L^q_1, L^q_2] = \frac{J^2}{2}\Delta L^q_3, \qquad [L^q_2, L^q_3] = -L^q_1, \qquad [L^q_3, L^q_1] = \frac{\Delta}{2} L^q_2 \tag{9}$$

Obviously, when $q \to 1$ they return to the common generating elements of $su(\eta(J))$, i.e., $L^q_i \to L_i$ ($i = 1, 2, 3$), where

$$L_1 = \frac{J^3}{2} K_3, \qquad L_2 = \frac{1}{2J}(K_+ + K_-), \qquad L_3 = \frac{1}{2}(K_+ - K_-)$$
$$[L_1, L_2] = J^2 L_3, \qquad [L_2, L_3] = -L_1, \qquad [L_3, L_1] = -L_2 \tag{10}$$

Note that the transformation from L^q_i to L_i ($i = 1, 2, 3$) is in fact an affine isomorphism,

$$L^q_1 = L_1, \qquad L^q_2 = \left(\frac{\Delta}{2}\right)^{1/2} L_2, \qquad L^q_3 = \left(\frac{\Delta}{2}\right)^{1/2} L_3 \tag{11}$$

3. THE q-DEFORMED ERNST EQUATION

Following Dodd and Morris (1982), we consider the nonconstant matrix M^q, $N^q \in su_q(\eta(J))$,

$$M^q = \alpha^q_1 L^q_1 + \beta^q_1 L^q_2 + \gamma^q_1 L^q_3$$
$$N^q = \alpha^q_2 L^q_1 + \beta^q_2 L^q_2 + \gamma^q_2 L^q_3 \tag{12}$$

where $\alpha^q_\tau = \alpha^q_\tau(x, y; q; J)$, $\beta^q_\tau = \beta^q_\tau(x, y; q; J)$, and $\gamma^q_\tau = \gamma^q_\tau(x, y; q; J)$ ($\tau = 1, 2$) are functions of the real variables x and y, and q is the deformation parameter. We consider a three-dimensional linear system which relates to $su_q(\eta(J))$ as follows:

$$\partial_1\phi^q = M^q\phi^q, \qquad \partial_2\phi^q = N^q\phi^q, \qquad \phi^q = \begin{bmatrix} \phi_1^q \\ \phi_2^q \\ \phi_3^q \end{bmatrix} \tag{13}$$

where $\partial_1 = \partial/\partial x$ and $\partial_2 = \partial/\partial y$, and $\phi_k^q = \phi_k^q(x, y; q; J)$ $(k = 1, 2, 3)$ are real functions. The integrability condition of equation (13) is

$$\partial_2 M^q - \partial_1 N^q + [M^q, N^q] = 0 \tag{14}$$

We write

$$c = c(x, y) = \frac{1 + v^2}{1 - v^2}, \qquad s = s(x, y) = \frac{2v}{1 - v^2} \tag{15}$$

where the real function $v = v(x, y)$ is a solution of the equations

$$(1 - v^2)\partial_1 v + \frac{1}{x}(1 + v^2)v = 0$$

$$(1 - v^2)\partial_2 v - \frac{1}{2x}(1 + v^2)^2 = 0 \tag{16}$$

The concrete form of v can be easily written out. We see that, in fact, c and s are the hyperbolic cosine and the hyperbolic sine functions, respectively, which obey the relation

$$c^2 - s^2 = 1 \tag{17}$$

Suppose that $P_\mu(x, y; q; J)$ and $Q_\mu(x, y; q; J)$ $(\mu = 1, 2)$ are real functions, and in equation (12) take α, β and γ as $\alpha_\tau^q = Q_\tau^q$ $(\tau = 1, 2)$, $\beta_\mu^q = sQ_\mu^q - cQ_\nu^q$, $\gamma_\mu^q = sP_\mu^q - cP_\nu^q$ $(\mu, \nu = 1, 2)$ (Dodd and Morris, 1982); then

$$M^q = Q_1^q L_1^q + (sQ_1^q - cQ_2^q)L_2^q + (sP_1^q - cP_2^q)L_3^q$$

$$N^q = Q_2^q L_1^q + (sQ_2^q - cQ_1^q)L_2^q + (sP_2^q - cP_1^q)L_3^q \tag{18}$$

and then from equation (14) we obtain

$$\left\{\begin{aligned}
&\partial_2 Q_1^q - \partial_1 Q_2^q - (sQ_1^q - cQ_2^q)(-cP_1^q + sP_2^q) \\
&\quad - (sP_1^q - cP_2^q)(cQ_1^q - sQ_2^q) = 0 && \text{(19a)} \\
&\partial_2(sQ_1^q - cQ_2^q) + \partial_1(cQ_1^q - sQ_2^q) \\
&\quad - \frac{\Delta}{2}[(sP_1 - cP_2)Q_2^q + (cP_1^q - sP_2^q)]Q_1^q = 0 && \text{(19b)} \\
&\partial_2(sP_1^q - cP_2^q) + \partial_1(cP_1^q - sP_2^q) \\
&\quad + \frac{J^2\Delta}{2}[(cQ_1^q - sQ_2^q)Q_1^q + (sQ_1^q - c_2Q_2^q)]Q_2^q = 0 && \text{(19c)}
\end{aligned}\right.$$

From (16) and (17), we can further change equations (19) into

$$\begin{cases} \partial_\mu P^q_\mu = \partial_\nu P^q_\mu, \quad \partial_\mu Q^q_\nu - \partial_\nu Q^q_\mu = P^q_\nu Q^q_\mu - P^q_\mu Q^q_\nu \\ \qquad (\mu, \nu = 1, 2) & (20a) \\ \partial_1 P^q_1 + \dfrac{1}{x} P^q_1 - \partial_2 P^q_2 = \dfrac{J^2\Delta}{2} [(Q^q_2)^2 - (Q^q_1)^2] & (20b) \\ \partial_1 Q^q_1 + \dfrac{1}{x} Q^q_1 - \partial_2 Q^q_2 = \dfrac{\Delta}{2} (P^q_1 Q^q_1 - P^q_2 Q^q_2) & (20c) \end{cases}$$

Now, we take two real functions $f^q(x, y; q)$ and $\psi^q(x, y; q)$, and

$$\partial_\mu f^q = P^q_\mu f^q, \qquad \partial_\mu \psi^q = Q^q_\mu f^q \qquad (\mu = 1, 2) \tag{21}$$

This is allowable since the integrability condition of equations (21) is just equations (20). Now, from equations (20b) and (20c) the integrability condition of system (13) is

$$\begin{cases} f^q \nabla^2_{(J)} f^q = \nabla_{(J)} f^q \cdot \nabla_{(J)} f^q + \dfrac{J^2\Delta}{2} \nabla_{(J)} \psi^q \nabla_{(J)} \psi^q \\ f^q \nabla^2_{(J)} \psi^q = \left(1 + \dfrac{\Delta}{2}\right) \nabla_{(J)} f^q \nabla_{(J)} \psi^q \end{cases} \tag{22}$$

where $\nabla^2_{(J)}$ and $\nabla_{(J)}$ are defined as

$$\nabla^2_{(J)} = \partial^2_x + J^2 \frac{1}{x} \partial_x + \partial^2_y, \qquad \nabla_{(J)} = (\partial_x, J\partial_y) \tag{23}$$

i.e.,

$$\nabla^2_{(i)} = \partial^2_x - \frac{1}{x} \partial_x + \partial^2_y, \qquad \nabla_{(i)} = (\partial_x, i\partial_y) \qquad (J = i) \tag{24a}$$

$$\nabla^2_{(\epsilon)} = \partial^2_x + \frac{1}{x} \partial_x + \partial^2_y, \qquad \nabla_{(\epsilon)} = (\partial_x, \epsilon\partial_y) \qquad (J = \epsilon) \tag{24b}$$

When $q \rightarrow 1$, equations (22) become

$$\begin{cases} \mathrm{Re}(\mathscr{E})\, \nabla^2_{(J)} \mathscr{E} = \nabla_{(J)} \mathscr{E} \cdot \nabla_{(J)} \mathscr{E} & (25) \\ \mathscr{E} = f + J\psi & (26) \end{cases}$$

This is just the double complex Ernst equation (Zhong 1985, 1988, 1989). The case $J = i$ describes the axisymmetric gravitational field (Ernst, 1968), and the case $J = \mathscr{E}$ describes cylindrical gravitational waves (Letelier, 1984). Therefore, equations (22) represent a double quantum deformation of the general Ernst equation.

4. THE SOLUTION

In order to obtain a solution of equations (22), we take an arbitrary harmonic function ϕ, $\nabla^2\phi = 0$, and assume that $f^q = f^q(\phi)$, $\psi^q = \psi^q(\phi)$. Then equations (22) become

$$\begin{cases} f^q\left(\dfrac{d^2f^q}{d\phi}\right)^2 = \left(\dfrac{df^q}{d\phi}\right)^2 + \dfrac{J^2\Delta}{2}\left(\dfrac{d\psi^q}{d\phi}\right)^2 & \text{(27a)} \\[2em] f^q\dfrac{d^2\psi^q}{d\phi^2} = \left(1 + \dfrac{\Delta}{2}\right)\dfrac{df^q}{d\phi}\cdot\dfrac{d\psi^q}{d\phi} & \text{(27b)} \end{cases}$$

From (27b), we have

$$\frac{1}{d\psi^q/d\phi}\, d\left(\frac{d\psi^q}{d\phi}\right) = \left(1 + \frac{\Delta}{2}\right)\frac{1}{f^q}\, df^q \tag{28}$$

From the integration of both sides of equation (28), we obtain

$$\frac{d\psi^q}{d\phi} = A_0(f^q)^{1+\Delta/2} \tag{29}$$

where A_0 is an integration constant. Substituting equation (29) into (27a) gives

$$\frac{d^2f^q}{d\phi^2} - \frac{1}{f^2}\left(\frac{df^q}{d\phi}\right)^2 = \frac{J^2\Delta}{2} A_0^2(f^q)^{1+\Delta} \tag{30}$$

We write $p = df^q/d\phi$; then equation (30) is

$$\frac{dp^2}{df^q} - \frac{2}{f^q}p^2 = J^2\Delta A_0^2(f^q)^{1+\Delta} \tag{31}$$

and its solution is

$$p = f^q + [c + J^2A_1^2(f^q)^\Delta]^{1/2} \tag{32}$$

where A_1 and c are integration constants. Therefore

$$d\phi = \frac{df^q}{f^q[c + J^2A_1^2(f^q)^\Delta]^{1/2}} \tag{33a}$$

$$\phi = \frac{c'}{\Delta}\ln\frac{c + [c + J^2A_1^2(f^q)^\Delta]^{1/2}}{[(f^q)^\Delta]^{1/2}} \tag{33b}$$

where c' is an integration constant.

From equation (33b), we can obtain a solution f^q in terms of ϕ,

$$f^q = \left[\frac{A(\phi)}{B(\phi)}\right]^{1/\Delta}$$

where

$$A(\phi) = c_0 e^{(\Delta/c_1)\phi} \pm [c_2 J^2 e^{(\Delta/c_1)\phi} + c_3 e^{(2\Delta/c_1)\phi}]^{1/2} + c_4 J^2$$
$$B(\phi) = (e^{(\Delta/c_1)\phi} - c_5 J^2)^2 \tag{34}$$

c_μ ($\mu = 1, 2, \ldots, 5$) are constants. According to equation (29),

$$\psi^q = A_0 \left[\frac{A(\phi)}{B(\phi)}\right]^{1/\Delta + 1/2} \phi + c \tag{35}$$

It is easily seen that if we take $A_0 = 0$ and $c_\mu = 0$ for $\mu = 2, \ldots, 5$, while $c_0 = 1$ and $c_1 = -1$, then

$$\begin{cases} f^q = e^\phi \\ \psi^q = 0 \end{cases} \tag{36}$$

This just becomes the well-known Weyl solution.

REFERENCES

Biedenharn, L. C. (1989). *Journal of Physics A: Mathematical and General Physics*, **22**, L873.
Dodd, R. K., and Morris, H. C. (1982). *Journal of Mathematical Physics*, **23**, 1131.
Ernst, F. J. (1968). *Physical Review*, **167**, 1175.
Letelier, P. S. (1984). *Journal of Mathematical Physics*, **25**, 2675.
Zhong, Z. Z. (1985). *Journal of Mathematical Physics*, **26**, 2589.
Zhong, Z. Z. (1988). *Scientia Sinica A*, **31**, 436.
Zhong, Z. Z. (1989). *Journal of Mathematical Physics*, **30**, 1158.
Zhong, Z. Z. (1992a). In *International Conference on Fluid Mechanics and Theoretical Physics*, p. 250.
Zhong, Z. Z. (1992b). *Journal of Physics A: Mathematical and General Physics*, **25**, L867.